# THE CONSTRUCTION OF SOME APPROXIMATE NON-SIMILARITY SOLUTIONS OF THE PROBLEM OF ONE-DIMENSIONAL UNSTEADY FILTRATION IN A POROUS MEDIUM

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*PMM Vol.24, No.6, 1960, pp. 1132-1135*

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*(Received July 21, 1960)*

We shall consider here a method for the construction of some non-similarity solutions of the equation of plane isothermal unsteady motion of a gas in a porous medium, based on the use of self-similar solutions for the same equation [1,2].

1. It is known that the solutions of the form $t^{\alpha} f(xt^{-\beta})$, $e^{\alpha x} f(te^{\beta x})$ and $e^{\alpha t} f(xe^{\beta t})$, where $\alpha$ and $\beta$ are constants satisfying certain relations, exhaust all the self-similar solutions of the equation. To these problems correspond the following boundary and initial dependences for the density $\rho(x, t)$ at $x = 0$ and $t = 0$ respectively:

$$\rho(0, t) = \sigma t^{\alpha} = \sigma e^{\alpha t}, \ \rho(x, 0) = \sigma e^{\alpha x} \tag{1.1}$$

Using some particular gas motions, those with similarity solutions, we can obtain a solution for cases in which the function $\rho(0, t)$ of the growth of the pressure at the zero section ($x = 0$) is a piecewise smooth continuous function.

In addition, the following should also be pointed out. From one point of view, the function $\rho(0, t)$ of the solution may be taken to be "exact" (the meaning of this concept will be clarified later); on the other hand, an estimate of $\rho(0, t)$ may be obtained from below or from above; in this case we can use the theorem of Barenblatt and Vishik on the monotone dependence of the solution of the differential equation of Boussinesq on initial and boundary conditions [3]; this can be generalized without difficulty to the larger general class of the differential equations which appear in gas and liquid filtration in a porous medium.

2. We will make use of some results of [2]. We limit ourselves to plane flows of a gas with fixed temperature. The equation in this case has the following form:

$$c\partial\rho / \partial t = \partial^2\rho^2 / \partial x^2 \qquad (2.1)$$

The solution of Equation (2.1) for boundary and initial conditions

$$\rho(0, t) = \sigma t^p \quad (p \geqslant 0), \qquad \rho(x, 0) = 0 \quad (x \geqslant 0) \qquad (2.2)$$

has the form

$$\rho(x, t) = \sigma t^p f(x\sigma^{-1/2} c^{1/2} t^{-(1+p)/2}) \qquad (2.3)$$

where $f(\xi)$ is determined by the following problem:

$$\left(\frac{df}{d\xi}\right)^2 + f\frac{d^2f}{d\xi^2} + \frac{p+1}{4}\xi\frac{df}{d\xi} - \frac{p}{2}f = 0 \qquad \left(f(0) = 1, \int_0^\infty f(\xi)\,d\xi < \infty\right) \qquad (2.4)$$

Reference [2] presents a table of the function $f(\xi, p)$ obtained as a result of integrating (2.4) with the help of power series. Our analysis of the function shows that for $3 \geqslant p \geqslant 0.5$ all curves may be replaced with great accuracy with broken lines

$$f(\xi) = 1 - \xi / \xi_0 \quad \text{for} \quad \xi \leqslant \xi_0, \qquad f(\xi) = 0 \quad \text{for} \quad \xi > \xi_0 \qquad (2.5)$$

where $\xi_0$ is the coordinate of the wave front, determined from the graph $\xi_0 = \xi(p)$ introduced in [2].

Thus, for $3 \geqslant p \geqslant 0.5$ the function $f(\xi)$ in (2.5) can be obtained with sufficient accuracy for all practical purposes for the construction of the solution of Equation (2.1).

We note that for the case $p = 1$ the solution $1 - \xi/\xi_0$, where $\xi_0 = \sqrt{2}$, is accurate in the exact sense.

It appears that the function $f(\xi)$ (2.5) can be very simply worked out for the solution for the case of the piecewise smooth function $\rho(0, t)$, constructed from sections of the following curves:

$$\rho(0, t) = \sigma_1 t^{p_1} \quad \text{for} \quad t_1 \geqslant t \geqslant 0, \qquad \rho(0, t) = \sigma_2 (t + \tau)^{p_2} \quad \text{for} \quad t \geqslant t_1 \qquad (2.6)$$

where $t_1$, $\tau$, $p_1$, $p_2$, $\sigma_1$ and $\sigma_2$ are constants connected (as we shall see below) by some relations. For that, it is evident that $3 \geqslant p_{1,2} \geqslant 0.5$.

The solution is obtained in the following form. For $t = t_1$ we have

$$\sigma_1 t_1^{p_1} = \sigma_2 (t_1 + \tau)^{p_2} \qquad (2.7)$$

At the time $t = t_1$ the density distribution, corresponding to the case

$\rho_1(0, t) = \sigma_1 t^{p_1}$, according to (2.5) has the following form:

$$\rho_1(x, t_1) = \sigma_1 t_1^{p_1}\left[1 - \frac{x}{\xi_0(p_1)(\sigma_1 c^{-1} t^{1+p_1})^{1/2}}\right] \tag{2.8}$$

At the same value of time but for the condition

$$\rho_2(0, t) = \sigma_2 (t + \tau)^{p_2} \tag{2.9}$$

the density distribution will be

$$\rho_2(x, t_1) = \sigma_2 (t_1 + \tau)^{p_2}\left[1 - \frac{x}{\xi_0(p_2)(\sigma_2 c^{-1}(t_1 + \tau)^{1+p_2})^{1/2}}\right] \tag{2.10}$$

We require at that value of time a density distribution which satisfies both (2.8) and (2.10). Thus, taking into account (2.9), we obtain

$$\xi_0(p_1)\sigma^{1/2} t^{(1+p_1)/2} = \xi_0(p_2)\sigma_2^{1/2}(t_1 + \tau)^{(1+p_2)/2} \tag{2.11}$$

For condition (2.9) and (2.11) superposition of the values $\sigma_1$, $\sigma_2$, $\tau$ and $t_1$ shows that Expressions (2.8) and (2.10) will obviously be "exact" solutions (in the sense considered above) of the problems corresponding, respectively, to the intervals $t_1 \geqslant t \geqslant 0$ and $t \geqslant t_1$.

*Example*. Let $\rho_1(0, t) = \sigma_1 t$, $\rho_2(0, t) = \sigma_2(t + \tau)^2$. In this case, from the graph of $\xi_0(p)$ in [3], $\xi_0(1) = \sqrt{2}$ and $\xi_0(2) = 1.1138$. Thus, from Formulas (2.9) and (2.11) we have

$$\sigma_2 = \sigma_1 \frac{t_1}{(t_1 + \tau)^2}, \qquad \sqrt{2}\,\sigma_1^{1/2} t_1 = 1.1138\,\sigma_2^{1/2}(t_1 + \tau)^{3/2}$$

From this $\tau = 0.613\ t_1$, $\sigma_2 = 0.384\ \sigma_1/t_1$. Thus, for a piecewise smooth continuous function of the density at $x = 0$ of the form

$$\rho(0, t) = \begin{cases} \sigma_1 t & (t_1 \geqslant t \geqslant 0) \\ 0.384(\sigma_1/t_1)(0.613\,t_1 + t)^2 & (t \geqslant t_1) \end{cases}$$

the solution has the following form:

$$\rho(x, t) = \begin{cases} \sigma_1 t\left[1 - \dfrac{c^{1/2}x}{\sqrt{2}\,\sigma_1^{1/2} t}\right] \\ 0.384\,\dfrac{\sigma_1}{t_1}(0.613\,t_1 + t)^2\left[1 - \dfrac{c^{1/2}x}{1.1138\,(0.384\,\sigma_1/t_1)^{1/2}(0.613\,t_1 + t)^{3/2}}\right] \end{cases}$$

**3.** In the preceding section we were able to obtain "exact" solutions for non-similarity cases because the power exponent $p$ corresponded to the linear character of the density distribution in the $x$-variable. Evidently less satisfactory results will be obtained for piecewise smooth continuous functions with exponent $p$ smaller than 0.5. However, actual density distributions, which do not require more exactness, with $\xi < \xi_0$ can be approximated [4] with a straight line of the form $1 - \xi/\xi_0$, and, as indicated above, the solution for piecewise smooth continuous functions can be extended to all values of the exponent from zero to three. However, it is next necessary to estimate "from below" and "from above" the solutions obtained by the method presented. It is especially important in those cases where the profiles of the density distribution to be studied have an essentially convex or concave form.

The idea of the estimate can be stated in the following way: suppose that for a power law of growth of density at the $x = 0$ $\rho(0, t) = \sigma_1 t^{p_1}$ there is obtained at $t = t_1$ the density distribution

$$\rho_1(x, t_1) = \sigma_1 t_1^{p_1} f_1(x\sigma_1^{-1/2} c^{1/2} t^{-(1+p_1)/2}) \tag{3.1}$$

For $t > t$ the law of density growth at $x = 0$ will be $\rho(0, t) = \sigma_2(\tau + t)^{p_2}$. To this law there corresponds for $t = t_1$ the following density distribution:

$$\rho_2(x, t) = \sigma_2(\tau + t_1)^{p_2} f_2(x\sigma_2^{-1/2} c^{1/2} (\tau + t_1)^{-(1+p_2)/2}) \tag{3.2}$$

where for no values of the parameters $\tau$, $\sigma_1$ and $\sigma_2$ of the functions $f_1$ and $f_2$ can the density distributions coincide.

Now consider two solutions $\rho^{(1)}(x, t)$, $\rho^{(2)}(x, t)$ of Equation (2.1), where $\rho^{(1)}(x, t_1) \geqslant \rho^{(2)}(x, t_1)$ for $t = t_1$ and for $t > t_1$, $\rho^{(1)}(0, t) \geqslant \rho^{(2)}(0, t)$.

We choose the parameters which appear in the boundary conditions for the solutions $\rho^{(1)}$ and $\rho^{(2)}$, in such a manner that

$$\rho^{(1)}(x, t_1) \geqslant \rho_{1,2}(x, t_1) \geqslant \rho^{(2)}(x, t_1), \qquad \rho^{(1)}(0, t) \geqslant \rho_{1,2}(0, t) \geqslant \rho^{(2)}(0, t) \tag{3.3}$$

Then, according to the theorem of Barenblatt and Vishik [3], the following inequality must hold:

$$\rho^{(1)}(x, t) \geqslant \rho_{1,2}(x, t) \geqslant \rho^{(2)}(x, t) \qquad \text{for } t > t_1 \tag{3.4}$$

The choice of the majorant $\rho^{(1)}$ and the minorant $\rho^{(2)}$ depends on the conditions of the problem.*

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* It should be noted that in [3] an indication is given of the possibility of using this theorem for estimating the solution of Equation (2.1).

*Example.* Let

$$\rho(0, t) = \sigma_1 t \quad \text{for } t_1 \geqslant t \geqslant 0, \qquad \rho(0, t) = \sigma_2 \quad \text{for } t \geqslant t_1 \tag{3.5}$$

For $t = t_1$ the density distribution has the form

$$\rho_1(x, t) = \sigma_1 t_1 \left[1 - \left(\frac{c}{2\sigma_1}\right)^{1/2} \frac{x}{t_1}\right] \tag{3.6}$$

For the same moment of time $t_1$ the density distribution corresponding to an instantaneous growth of density at the zero section [$p = 0$] at a time $\tau^{(1)}$ up to the value $\sigma_2 = \sigma_1 t_1$ will be

$$\rho_2(x, t) = 5.22\sigma_2 \left\{\frac{1}{4}[1 - 0.4375\xi_1] - \frac{1}{16}[1 - 0.4375\xi_1]^2 + \ldots\right\} \tag{3.7}$$

$$\xi_1 = x\sigma_2^{-1/2} c^{1/2} (t_1 - \tau^{(1)})^{-1/2} \tag{3.8}$$

In the figure the line labeled 1 and the curve 2 are contructed according to Formulas (3.6) and (3.7), where the interval of time $\tau^{(1)}$ is chosen so that the equality condition at the gas front for both cases is satisfied, that is

$$\xi_0(1)\sigma_1^{1/2} t_1 = \xi_0(0)\sigma_2^{1/2}(t_1 - \tau^{(1)})^{1/2} \tag{3.9}$$

From here, with the substitution $\xi_0(1) = \sqrt{2}$, $\xi_0(0) = 2.2857$ and $\sigma_2 = \sigma_1 t_1$, we have $\tau^{(1)} = 0.617\ t_1$.

In this case it is appropriate to choose (3.7) as majorant $\rho^{(1)}$, that is, $\rho^{(1)}(x, t) = \rho_2(x, t)$, where $t \geqslant t_1$ and for $\rho^2$ a function corresponding to an instantaneous growth of the density at the zero section to a certain value $\kappa\sigma_2 (\kappa < 1)$ for $t = \tau^{(2)}$; it is necessary to choose $\kappa$ and $\tau^{(2)}$ in such a way that the whole curve 3 of the density distribution at the moment $t = t_1$ lies under the line 1 and the tangent to it (see figure).

By graphical means it was found that $\kappa \approx 0.93$, $\tau^{(1)} \approx 0.645\ t_1$.

Thus, $\rho^{(1)}$ and $\rho^{(2)}$ must have the following form:

$$\rho^{(1)} = 5.22\sigma_1 t_1 \left\{\frac{1}{4}[1 - 0.4375\xi^{(1)}] - \frac{1}{16}[1 - 0.4375\xi^{(1)}]^2 + \ldots\right\}$$

$$\xi^{(1)} = x(\sigma_1 t_1)^{-1/2} c^{1/2} [t - 0.617 t_1]^{-1/2}$$

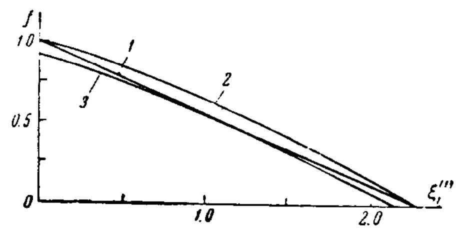
$$\rho^{(2)} = 4.85\sigma_1 t_1 \left\{\frac{1}{4}[1 - 0.4375\xi^{(2)}] - \frac{1}{16}[1 - 0.4375\xi^{(2)}]^2 + \ldots\right\}$$

$$\xi^{(2)} = x(0.93\sigma_1 t_1)^{-1/2} c^{1/2} [t - 0.645 t_1]^{-1/2}$$

and condition (3.4) for the function $\rho_2(x, t)$ for $t > t_1$ will be satisfied.

The area of the strip contained between 2 and 3 in the figure is roughly 15 per cent of the area lying below curve 2.

From this it follows that $Q^{(1)}/Q^{(2)} \approx 1.15$, where $Q^{(1)}$ and $Q^{(2)}$ are volumes of fluid entering the porous medium at the time $t = t_1$ corresponding to $\rho^{(1)}$ and $\rho^{(2)}$. Therefore, when calculating the volume of the fluid in any case, the bounded variant in the error must not be greater than 15 per cent. From this it follows that the relative error is of order 7 to 8 per cent.




We observe that

$$Q^{(1)} / Q^{(2)} \to \varkappa^{-1/3} \approx 1.1 \quad \text{for } t \to \infty$$

In a completely analogous manner problems can also be solved for the more common case

$$\rho(0, t) = \begin{cases} \sigma_1 t & (t_1 \geqslant t \geqslant 0) \\ \sigma_2 (t + \tau)^{p_2} & (t \geqslant t_1,\ p_2 > 0) \end{cases}$$

We do not carry out the calculation in detail here. We observe only that $p_2 = 0.25$, $Q^{(1)}/Q^{(2)} \approx 1.08$ for $t = t_1$, that is, the error rectilinearly approximated will be of the order of 4 per cent.

4. It is known that the similarity solution of Equation (2.1)

$$\rho(x, t) = \rho_0 e^{\sigma t} f\left(\frac{x}{\sqrt{c^{-1}\rho_0 \sigma^{-1} e^{\sigma t}}}\right) \tag{4.1}$$

which corresponds to the exponential law of growth $\rho(0, t) = \rho_0 \exp(\sigma t)$ at $x = 0$ [4 ], can be approximated by the straight line

$$f(\xi) = 1 - \xi / \xi_0$$

Evidently it is possible on the basis of this to obtain the solution for the case in which $\rho(0, t)$ is a piecewise smooth function consisting of an exponential and power dependency.

Let the density at $x = 0$, $0 < t < t_1$ vary according to a power law; for $t > t_1$ the density behaves like the exponential $\rho(0, t) = \rho_0 \exp(\sigma t)$.

We consider the solutions $\rho^{(1)}$ and $\rho^{(2)}$ corresponding to the following law of growth of the density at $x = 0$:

$$\rho^{(1)}(0,t) = \rho_0 e^{\sigma t} \rho^{(2)}(0,t) = \varkappa \rho_0 e^{\sigma t} \qquad (\varkappa < 1)$$

We choose $\kappa$ so that the density distribution at the end of the first stage is contained in a strip of the distribution $\rho^{(1)}(x, t_1)$ and $\rho^{(2)}(x, t_1)$; thus, according to [3], relations (3.3) are satisfied.

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*Translated by* H.C.